Advanced Engineering Mathematics II and Differential Equations

Differentiation Rules

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}[cf(x)] = cf'(x)$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$	$\frac{d}{dx}f(g(x)) = \frac{d}{du}f(u) \cdot \frac{d}{dx}u$ where $u = g(x)$
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}e^x = e^x$
$\frac{d}{dx}a^x = a^x \ln a$	$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$
$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$	$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\sec x = \sec x \tan x$
$\frac{d}{dx}\cot x = -\csc^2 x$	

Table of Indefinite Integrals

$\int cf(x)dx = c\int f(x)dx$	$\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
$\int x^n \; dx = \frac{x^{n+1}}{n+1} + C \;\; (n \neq -1)$	$\int \frac{1}{x} \; dx = \ln\lvert x\rvert + C$
$\int e^x \; dx = e^x + C$	$\int \cos x \; dx = \sin x + C$
$\int \sin x \; dx = -\cos x + C$	$\int \sec^2 x \; dx = \tan x + C$
$\int \sec x \tan x \; dx = \sec x + C$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$	$\int \csc^2 x \; dx = -\cot x + C$
$\int \csc x \cot x \; dx = -\csc x + C$	$\int \sec x \; dx = \ln\lvert\sec x + \tan x\rvert + C$
$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$	$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{1}{a}\right) + C$

A second-order homogeneous equations with constant coefficients is an equation of the form

(0.1) $$y'' + ay' + by = 0.$$

The characteristic equation is

(0.2) $$\lambda^2 + a\lambda + b = 0.$$

Case	Roots of (0.2)	General Solution of (0.1)
I	Distinct real λ_1, λ_2	$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$y = (c_1 + c_2x)e^{-\frac{ax}{2}}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$y = e^{-\frac{ax}{2}}(A\cos\omega x + B\sin\omega x)$

An Euler-Cauchy equation is an equation of the form

$$x^2y'' + axy' + by = 0. \tag{0.3}$$

The auxiliary equation is

$$m^2 + (a-1)m + b = 0. \tag{0.4}$$

Case	Roots of (0.4)	General Solution of (0.3)
I	Distinct real m_1, m_2	$y = c_1x^{m_1} + c_2x^{m_2}$
II	Real Double root $m = \frac{1}{2}(1-a)$	$y = (c_1 + c_2 \ln x)x^{(1-a)/2}$
III	Complex conjugate $m_1 = \mu + i\nu$, $m_2 = \mu - i\nu$	$y = x^{\mu}[A\cos(\nu \ln x) + B\sin(\nu \ln x)]$

Table of Undetermined Coefficients

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, ...)$	$K_nx^n + K_{n-1}x^{n-1} + ... + K_1x + K_0$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Solution by Variation of Parameters

This method applies to differential equations

$$y'' + p(x)y' + q(x)y = r(x) \tag{0.5}$$

with arbitrary variable function p, q, and r that are continuous on some interval I. The method gives a particular solution y_p of (0.5) in the form

$$y_p(x) = -y_1\int\frac{y_2r}{W}dx + y_2\int\frac{y_1r}{W}dx \tag{0.6}$$

where y_1, y_2 form a basis of solution of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ and

$$W = y_1y_2' - y_2y_1'.$$

Useful Formulas

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - .... - f^{(n-1)}(0)$ where $f^{(n)} = \dfrac{d^n f}{dx^n}$.

$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \dfrac{1}{s}F(s)$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

$\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$

$\cosh at = \dfrac{1}{2}(e^{at} + e^{-at})$

$\sinh at = \dfrac{1}{2}(e^{at} - e^{-at})$

$\sin x \sin y = \dfrac{1}{2}[-\cos(x+y) + \cos(x-y)]$

$\cos x \cos y = \dfrac{1}{2}[\cos(x+y) + \cos(x-y)]$

$\sin x \cos y = \dfrac{1}{2}[\sin(x+y) + \sin(x-y)]$

Some functions $f(t)$ and their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, ...)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

Partial Fractions

Case	Partial Fractions
Unrepeated factors $s-a$	$\frac{A}{s-a}$
Repeated factors $(s-a)^m$	$\frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + ... + \frac{A_1}{s-a}$
Complex factors $(s-a)(s-\bar{a})$	$\frac{As+B}{(s-a)(s-\bar{a})}$
Repeated complex factors $[(s-a)(s-\bar{a})]^2$	$\frac{As+B}{[(s-a)(s-\bar{a})]^2} + \frac{Ms+N}{(s-a)(s-\bar{a})}$